Berkeley Evaluation & Assessment Research Center
Graduate School of Education
University of California, Berkeley
Berkeley, CA 94720-1670
http://bearcenter.berkeley.edu
Technical Paper Series
No. 04.10.02

# Interpreting Ordered Partition Model Parameters from ConQuest

Nathaniel J. S. Brown

October 2004

## Abstract

The ordered partition model (OPM) described in Wilson (1992) is an extension of Masters' (1982) partial credit model (PCM) that allows more than one response category to have a particular score. The generalized Rasch modeling software ConQuest (Wu, Adams, & Wilson, 1998) provides parameter estimates for the OPM, but using a different parameterization than given in Wilson (1992), complicating interpretation of those estimates. This report describes and illustrates the conversion of ConQuest OPM parameters into the Andersen level parameters for the OPM and PCM described in Wilson (1992), and into ConQuest PCM parameters.

## Introduction

Wilson (1992) describes the ordered partition model (OPM) as an extension of Masters' (1982) partial credit model (PCM) that breaks the one-to-one correspondence between response categories and scores, allowing more than one response to have a particular score. The OPM is useful for situations in which certain responses represent different but equally valued strategies or types of reasoning. In equations (25), (26), and (27), and the accompanying discussion, Wilson (1992) provides means of interpreting the parameters of the OPM, relating them to the relative probability of achieving particular response categories.

As a submodel of the random coefficients multinomial logit (RCML) model (Adams & Wilson, 1995), the OPM can easily be estimated using the `score` statement in ConQuest (Wu, Adams, & Wilson, 1998). However, while the discussion in Wilson (1992) is based upon Andersen level parameters, ConQuest uses item and step parameters, preventing the direct application of the interpretive strategies discussed in Wilson (1992). Moreover, because of the item-step parameterization, ConQuest OPM parameters cannot be directly compared to ConQuest PCM parameters, despite the fact that the two models are hierarchically related.

In the interest of facilitating the use of ConQuest in the estimation of the OPM, this report describes and illustrates the conversion of ConQuest OPM parameters into the Andersen level parameters for the OPM and PCM described in Wilson (1992), and into ConQuest PCM parameters. First, all four sets of parameters are defined using a common notation. Then, equations are presented that effect the sequential conversion of the ConQuest OPM parameters into their Andersen OPM, Andersen PCM, and ConQuest PCM parameterizations, as well as a direct conversion from the ConQuest OPM to the ConQuest PCM parameters.

## Model Parameterization

Consider $I$ items ($i$ = 1, …, $I$). Each item has $K_i$ + 1 response categories ($k$ = 0, …, $K_i$), which are graded into $M_i$ + 1 possible score levels ($m$ = 0, …, $M_i$). For item $i$, response $k$ is assigned to level $m$ by the scoring function $B_i(k)$, so that $B_i(k) = m$.

### ConQuest OPM parameters

ConQuest parameterizes the OPM using an item parameter and $K_i$ – 1 step parameters. For this parameterization, the item response probability model is:

$$P(X_{ni} = k) = \frac{\exp\left[\theta_n B_i(k) - \xi_i B_i(k) - \sum_{j=0}^{k} \xi_{ij}\right]}{\sum_{h=1}^{K_i} \exp\left[\theta_n B_i(h) - \xi_i B_i(h) - \sum_{j=0}^{h} \xi_{ij}\right]} \tag{1}$$

where $X_{ni}$ is a random variable that represents the response of person $n$ with ability $\theta_n$ to item $i$;

$\xi_i$ is an item parameter for item $i$;

$\xi_{ik}$ is a step parameter for item $i$ associated with reaching category $k$ from $k$ – 1;

$\xi_{i0} \equiv 0$;

and $\sum_{j=0}^{K_i} \xi_{ij} \equiv 0$.

### Andersen OPM parameters

Wilson (1992) parameterizes the OPM using $K_i$ Andersen level parameters. For this parameterization, the item response probability model is:

$$P(X_{ni} = k) = \frac{\exp\left[\theta_n B_i(k) - \eta_{ik}\right]}{\sum_{h=0}^{K_i} \exp\left[\theta_n B_i(h) - \eta_{ih}\right]} \tag{2}$$

where $X_{ni}$ is a random variable that represents the response of person $n$, with ability $\theta_n$, to item $i$;
$\eta_{ik}$ is a level parameter for item $i$ associated with category $k$;
and $\eta_{i0} \equiv 0$.

The model in (2) is equivalent to (21) in Wilson (1992), where the symbol $\eta$ replaces $\xi$ and the category index $k$ begins with 0 instead of 1.

### Andersen PCM parameters

Wilson (1992) parameterizes the PCM using $M_i$ Andersen level parameters. For this parameterization, the item response probability model is:

$$P(X_{ni} = m) = \frac{\exp\left[m\theta_n - \kappa_{im}\right]}{\sum_{h=0}^{M_i} \exp\left[h\theta_n - \kappa_{ih}\right]} \tag{3}$$

where $X_{ni}$ is a random variable that represents the response of person $n$, with ability $\theta_n$, to item $i$;
$\kappa_{im}$ is a level parameter for item $i$ associated with score $m$;
and $\kappa_{i0} \equiv 0$.

The model in (3) is equivalent to (3) in Wilson (1992), where the symbol $\kappa$ replaces $\eta$.

### ConQuest PCM parameters

ConQuest parameterizes the PCM using an item parameter and $M_i$ – 1 step parameters. For this parameterization, the item response probability model is:

$$P(X_{ni} = m) = \frac{\exp\left[m\theta_n - m\delta_i - \sum_{j=0}^{m}\delta_{ij}\right]}{\sum_{h=0}^{M_i}\exp\left[h\theta_n - h\delta_i - \sum_{j=0}^{h}\delta_{ij}\right]} \tag{4}$$

where $X_{ni}$ is a random variable that represents the response of person $n$, with ability $\theta_n$, to item $i$;

$\delta_i$ is an item parameter for item $i$;

$\delta_{im}$ is a step parameter for item $i$ associated with reaching level $m$ from $m - 1$;

$\delta_{i0} \equiv 0$;

and $\sum_{j=0}^{M_i}\delta_{ij} \equiv 0$.

## Parameter Conversion

### ConQuest OPM parameters to Andersen OPM parameters

Comparing (1) and (2), which must be equivalent, reveals:

$$\eta_{ik} = \xi_i B_i(k) + \sum_{j=0}^{k}\xi_{ij} \tag{5}$$

where the identifying constraints $\xi_{i0} \equiv 0$ and $\sum_{j=0}^{K_i}\xi_{ij} \equiv 0$ should be kept in mind.

*Example*. ConQuest gave the following output for an item with 18 response categories ($K_i = 17$) assigned to 5 score levels ($M_i = 4$), so that:

$B_i(0) = 0$,

$B_i(1) = 1$,

$B_i(2) = B_i(3) = B_i(4) = B_i(5) = B_i(6) = B_i(7) = B_i(8) = 2$,

$B_i(9) = B_i(10) = B_i(11) = B_i(12) = B_i(13) = B_i(14) = B_i(15) = 3$, and

$B_i(16) = B_i(17) = 4$.

```
================================================================================
Ordered Partition Model
TABLES OF RESPONSE MODEL PARAMETER ESTIMATES
================================================================================
TERM 1: item
--------------------------------------------------------------------------------
   VARIABLES                                UNWGHTED FIT    WGHTED FIT
---------------                             -------------  -------------
     item            ESTIMATE  ERROR      MNSQ     T        MNSQ    T
--------------------------------------------------------------------------------
 ...
 2 2                   0.190   0.140     0.98  -0.1      0.96  -0.2
 ...
================================================================================
TERM 2: item*step
--------------------------------------------------------------------------------
          VARIABLES                                  UNWGHTED FIT   WGHTED FIT
-----------------------                             -------------  -------------
  item        step           ESTIMATE  ERROR      MNSQ     T      MNSQ    T
--------------------------------------------------------------------------------
 ...
 2 2          1                -1.745   0.525     1.50   3.1      1.00   0.2
 2 2          2                -0.426   0.323     0.97  -0.2      1.00   0.1
 2 2          3                -0.405   0.260     0.95  -0.3      0.96  -0.2
 2 2          4                 0.693   0.223     0.93  -0.5      0.94  -0.7
 2 2          5                 0.182   0.215     0.93  -0.4      0.94  -0.8
 2 2          6                -0.470   0.212     0.94  -0.4      0.95  -0.8
 2 2          7                 0.694   0.212     0.91  -0.6      0.94  -1.1
 2 2          8                 1.390   0.214     0.92  -0.6      0.95  -0.9
 2 2          9                -0.138   0.215     0.92  -0.6      0.95  -0.9
 2 2          10                0.035   0.216     0.91  -0.6      0.94  -1.0
 2 2          11               -0.015   0.216     0.87  -0.9      0.91  -1.4
 2 2          12               -2.398   0.216     0.87  -0.9      0.91  -1.3
 2 2          13               -0.647   0.230     0.90  -0.7      0.94  -0.6
 2 2          14                3.045   0.396     0.62  -3.0      0.89  -0.2
 2 2          15               -1.386   0.425     0.61  -3.2      0.89  -0.2
 2 2          16                0.895   0.722     0.65  -2.7      0.94   0.1
 ...
```

Using (5) to calculate the Andersen OPM parameters gives:

$$\eta_{i0} = \xi_i B_i(0) + \xi_{i0} = 0.190(0) + 0 = 0$$

$$\eta_{i1} = \xi_i B_i(1) + \xi_{i0} + \xi_{i1} = 0.190(1) + 0 + (-1.745) = -1.555$$

...

$$\eta_{i16} = \xi_i B_i(16) + \xi_{i0} + \xi_{i1} + \cdots + \xi_{i16} = 0.190(4) + 0 + (-1.745) + \cdots + 0.895 = 0.064$$

$$\eta_{i17} = \xi_i B_i(17) + \sum_{j=0}^{17} \xi_{ij} = 0.190(4) + 0 = 0.760$$

where the definitions $\xi_{i0} \equiv 0$ and $\sum_{j=0}^{K_i} \xi_{ij} \equiv 0$ have been used.

These parameters can be used to interpret the results of the OPM analysis, following the discussion and equations (25), (26), and (27) in Wilson (1992).

**Andersen OPM parameters to Andersen PCM parameters**

Following the discussion and equation (22) in Wilson (1992), the Andersen OPM parameters can be converted into the equivalent Andersen PCM parameters using:

$$\kappa_{im} = \ln\left[\frac{\sum_{B_i(t)=m-1}\exp(-\eta_{it})}{\sum_{B_i(t)=m}\exp(-\eta_{it})}\right] \tag{6}$$

keeping in mind that $\kappa_{i0} \equiv 0$.

*Example*. Continuing to use the data from above, (6) gives:

$$\kappa_{i1} = \ln\left[\frac{\sum_{B_i(t)=0}\exp(-\eta_{it})}{\sum_{B_i(t)=1}\exp(-\eta_{it})}\right] = \ln\left[\frac{\exp(-\eta_{i0})}{\exp(-\eta_{i1})}\right] = \ln\left[\frac{\exp(0)}{\exp(1.555)}\right] = -1.555$$

where there is only one addend in both the numerator and denominator because there is only one category in each of the first two score levels.

**Andersen PCM parameters to ConQuest PCM parameters**

The ConQuest item parameter is the average of the Andersen level parameters:

$$\delta_i = \frac{1}{M_i}\sum_{h=1}^{M_i}\kappa_{ih} \tag{7}$$

and the ConQuest step parameters are the Andersen level parameters adjusted by this average:

$$\delta_{im} = \kappa_{im} - \delta_i \tag{8}$$

**ConQuest OPM parameters to ConQuest PCM parameters**

Alternatively, the ConQuest PCM parameters can be determined directly from the ConQuest OPM parameters using:

$$\delta_i = \xi_i + \frac{1}{M_i}\ln\left[\frac{\sum_{B_i(t)=0}\exp\left(-\sum_{j=0}^{t}\xi_{ij}\right)}{\sum_{B_i(t)=M_i}\exp\left(-\sum_{j=0}^{t}\xi_{ij}\right)}\right] \tag{9}$$

and

$$\delta_{im} = \xi_i - \delta_i + \ln\left[\frac{\sum_{B_i(t)=m-1}\exp\left(-\sum_{j=0}^{t}\xi_{ij}\right)}{\sum_{B_i(t)=m}\exp\left(-\sum_{j=0}^{t}\xi_{ij}\right)}\right] \tag{10}$$

which are derived in the Appendix.

Note that (9) implies that, in most cases, the item parameters for the ConQuest OPM and PCM will not be equivalent. Only for the special case that there is a single category in both the lowest ($m = 0$) and highest ($m = M_i$) score levels will (9) simplify to:

$$\delta_i = \xi_i + \frac{1}{M_i}\ln\left[\frac{\exp\left(-\sum_{j=0}^{0}\xi_{ij}\right)}{\exp\left(-\sum_{j=0}^{K_i}\xi_{ij}\right)}\right] = \xi_i + \frac{1}{M_i}\ln\left[\frac{\exp(0)}{\exp(0)}\right] = \xi_i + \frac{1}{M_i}\ln(1) = \xi_i \tag{11}$$

resulting in equal item parameters for the ConQuest OPM and PCM.

*Example*. Continuing to use the data from above, (9) gives:

$$\delta_i = \xi_i + \frac{1}{4}\ln\left[\frac{\exp\left(-\sum_{j=0}^{0}\xi_{ij}\right)}{\exp\left(-\sum_{j=0}^{16}\xi_{ij}\right)+\exp\left(-\sum_{j=0}^{17}\xi_{ij}\right)}\right] = 0.190 + \frac{1}{4}\ln\left[\frac{\exp(0)}{\exp(0.696)+\exp(0)}\right] = -0.085$$

where the definitions $\xi_{i0} \equiv 0$ and $\sum_{j=0}^{K_i}\xi_{ij} \equiv 0$ have been used. The value of $\delta_i$ determined by ConQuest running the same data using the PCM was –0.086.

Using (10) gives:

$$\delta_{i1} = \xi_i - \delta_i + \ln\left[\frac{\exp\left(-\sum_{j=0}^{0}\xi_{ij}\right)}{\exp\left(-\sum_{j=0}^{1}\xi_{ij}\right)}\right] = 0.190 - (-0.085) + \ln\left[\frac{\exp(0)}{\exp(1.745)}\right] = -1.470$$

where the definition $\xi_{i0} \equiv 0$ has been used. The value of $\delta_{i1}$ determined by ConQuest running the same data using the PCM was –1.471. In general, the ConQuest PCM parameters were fully recoverable from the ConQuest OPM parameters in this example.

## References

Adams, R. J. & Wilson, M. (1995). Formulating the Rasch model as a mixed coefficients multinomial logit. In G. Engelhard & M. Wilson (Eds.), *Objective measurement: Theory into practice* (Vol. 3, pp. 143-166). Norwood, NJ: Ablex Publishing.

Masters, G. N. (1982). A Rasch model for partial credit scoring. *Psychometrika*, **47**, 149-174.

Wilson, M. (1992). The ordered partition model: An extension of the partial credit model. *Applied Psychological Measurement*, **16**, 309-325.

Wu, M. L., Adams, R. J., & Wilson, M. (1998). *ACERConQuest user guide*. Hawthorn, Australia: ACER Press.

## Appendix

The derivation of (9) and (10) follows. Substituting (5) into (6) gives:

$$\kappa_{im} = \ln\left[\frac{\sum_{B_i(t)=m-1} \exp\left(-\xi_i B_i(t) - \sum_{j=0}^{t} \xi_{ij}\right)}{\sum_{B_i(t)=m} \exp\left(-\xi_i B_i(t) - \sum_{j=0}^{t} \xi_{ij}\right)}\right] \tag{12}$$

which simplifies:

$$\kappa_{im} = \ln\left[\frac{\sum_{B_i(t)=m-1} \exp\left(-(m-1)\xi_i - \sum_{j=0}^{t} \xi_{ij}\right)}{\sum_{B_i(t)=m} \exp\left(-m\xi_i - \sum_{j=0}^{t} \xi_{ij}\right)}\right] \tag{13}$$

$$\kappa_{im} = \ln\left[\frac{\exp(-(m-1)\xi_i) \sum_{B_i(t)=m-1} \exp\left(-\sum_{j=0}^{t} \xi_{ij}\right)}{\exp(-m\xi_i) \sum_{B_i(t)=m} \exp\left(-\sum_{j=0}^{t} \xi_{ij}\right)}\right] \tag{14}$$

$$\kappa_{im} = \ln\left[\exp(\xi_i) \frac{\sum_{B_i(t)=m-1} \exp\left(-\sum_{j=0}^{t} \xi_{ij}\right)}{\sum_{B_i(t)=m} \exp\left(-\sum_{j=0}^{t} \xi_{ij}\right)}\right] \tag{15}$$

$$\kappa_{im} = \xi_i + \ln\left[\frac{\sum_{B_i(t)=m-1} \exp\left(-\sum_{j=0}^{t} \xi_{ij}\right)}{\sum_{B_i(t)=m} \exp\left(-\sum_{j=0}^{t} \xi_{ij}\right)}\right] \tag{16}$$

Plugging (16) into (7) gives:

$$\delta_i = \frac{1}{M_i}\sum_{h=1}^{M_i}\left[\xi_i + \ln\left[\frac{\sum_{B_i(t)=h-1}\exp\left(-\sum_{j=0}^{t}\xi_{ij}\right)}{\sum_{B_i(t)=h}\exp\left(-\sum_{j=0}^{t}\xi_{ij}\right)}\right]\right] \tag{17}$$

$$\delta_i = \xi_i + \frac{1}{M_i}\sum_{h=1}^{M_i}\ln\left[\frac{\sum_{B_i(t)=h-1}\exp\left(-\sum_{j=0}^{t}\xi_{ij}\right)}{\sum_{B_i(t)=h}\exp\left(-\sum_{j=0}^{t}\xi_{ij}\right)}\right] \tag{18}$$

$$\delta_i = \xi_i + \frac{1}{M_i}\ln\left[\prod_{h=1}^{M_i}\frac{\sum_{B_i(t)=h-1}\exp\left(-\sum_{j=0}^{t}\xi_{ij}\right)}{\sum_{B_i(t)=h}\exp\left(-\sum_{j=0}^{t}\xi_{ij}\right)}\right] \tag{19}$$

$$\delta_i = \xi_i + \frac{1}{M_i}\ln\left[\frac{\sum_{B_i(t)=0}\exp\left(-\sum_{j=0}^{t}\xi_{ij}\right)}{\sum_{B_i(t)=M_i}\exp\left(-\sum_{j=0}^{t}\xi_{ij}\right)}\right] \tag{9}$$

Likewise, plugging (16) into (8) gives (10) directly.