Solutions to the exercises of Chapter 3

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- Exercise 3.1
  The fixed-effects predictor matrix for the PCM can be written as (suppose there are three items each having three categories):

Item	Logit	X_1^*	X_2^*	X_3^*	X_4^*	X_5^*	X_6^*
1	L_1	1	0	0	1	1	0
1	L_2	1	0	0	0	0	0
2	L_1	0	1	0	1	0	1
2	L_2	0	1	0	0	0	0
3	L_1	0	0	1	1	0	0
3	L_2	0	0	1	0	0	0

  where X_1^*, X_2^*, and X_3^* are the item main effect predictors, X_4^* the logit main effect predictor (the predictor coding for the second logit is left out to ensure identifiability) and X_5^* and X_6^* are interaction predictors formed by multiplying X_4^* with X_1^* and X_2^*, respectively. Also including the product between X_4^* and X_3^* would result in a non-identifiable model. These predictors can also be found from Table 3.1 as follows:

$$\begin{aligned} X_1^* &= X_1 + X_2 \\ X_2^* &= X_3 + X_4 \\ X_3^* &= X_5 + X_6 \\ X_4^* &= X_1 + X_3 + X_5 \\ X_5^* &= X_1 \\ X_6^* &= X_2. \end{aligned}$$

  Then leaving out the interaction predictors gives the RSM predictor matrix.

- Exercise 3.2
  In this context, it is not unreasonable to consider a rating scale model. The fixed-effects predictor matrix includes five predictors: two item indicator predictors (X_1 and X_2), two logit indicator predictors (X_3 and X_4) and a fifth predictor (X_5) that is included to test the hypothesis of interest. Predictor X_5 can only be nonzero for males when one of the extreme

categories appears in the logit: for the first logit (comparing responding in category "never" vs. "rarely"), its value is -1, it is zero for the second logit (because no extreme category is involved) and it is 1 for the third logit (comparing "often" vs. "always"). The fixed-effects predictor matrix is then defined as follows:

Gender	Item	Logit	X_1	X_2	X_3	X_4	X_5
F	1	L_1	1	0	0	0	0
F	1	L_2	1	0	1	0	0
F	1	L_3	1	0	0	1	0
F	2	L_1	0	1	0	0	0
F	2	L_2	0	1	1	0	0
F	2	L_3	0	1	0	1	0
M	1	L_1	1	0	0	0	-1
M	1	L_2	1	0	1	0	0
M	1	L_3	1	0	0	1	1
M	2	L_1	0	1	0	0	-1
M	2	L_2	0	1	1	0	0
M	2	L_3	0	1	0	1	1

Let us denote the parameters corresponding to the five predictors as β_1, β_2, λ_1, λ_2 and δ, respectively. The four category probabilities for a male respondent p look as follows:

$$
\begin{aligned}
\pi_{pi0} &= \frac{1}{1+\exp(\theta_p-\beta_i+\delta)+\exp(2\theta_p-2\beta_i-\lambda_1+\delta)+\exp(3\theta_p-3\beta_i-\lambda_1-\lambda_2)} \\
\pi_{pi1} &= \frac{\exp(\theta_p-\beta_i+\delta)}{1+\exp(\theta_p-\beta_i+\delta)+\exp(2\theta_p-2\beta_i-\lambda_1+\delta)+\exp(3\theta_p-3\beta_i-\lambda_1-\lambda_2)} \\
\pi_{pi2} &= \frac{\exp(2\theta_p-2\beta_i-\lambda_1+\delta)}{1+\exp(\theta_p-\beta_i+\delta)+\exp(2\theta_p-2\beta_i-\lambda_1+\delta)+\exp(3\theta_p-3\beta_i-\lambda_1-\lambda_2)} \\
\pi_{pi3} &= \frac{\exp(3\theta_p-3\beta_i-\lambda_1-\lambda_2)}{1+\exp(\theta_p-\beta_i+\delta)+\exp(2\theta_p-2\beta_i-\lambda_1+\delta)+\exp(3\theta_p-3\beta_i-\lambda_1-\lambda_2)}
\end{aligned}
$$

If δ is positive, the odds of responding in any middle category compared to an extreme category increases for male respondents.

- Exercise 3.3
  In full item-by-logit model, the fixed-effect predictor matrix will look as follows (for three items):

Item	Logit	X_1	X_2	X_3	X_4	X_5	X_6	X_7
1	L_1	1	0	0	0	0	0	1
1	L_2	0	1	0	0	0	0	1
2	L_1	0	0	1	0	0	0	1
2	L_2	0	0	0	1	0	0	1
3	L_1	0	0	0	0	1	0	1
3	L_2	0	0	0	0	0	1	1

where the predictor X_7 represents the differential cost of responding in a higher category (compared to the one below). As one can see, $X_7 = X_1 + X_2 + X_3 + X_4 + X_5 + X_6$ and thus the predictor matrix contains linear dependencies so that not all parameters of these models can be estimated separately.

For the item and logit main-effects model, the situation is similar. The fixed-effects predictor matrix is the following:

Item	Logit	X_1	X_2	X_3	X_4	X_5
1	L_1	1	0	0	0	1
1	L_2	1	0	0	1	1
2	L_1	0	1	0	0	1
2	L_2	0	1	0	1	1
3	L_1	0	0	1	0	1
3	L_2	0	0	1	1	1

where the predictor X_5 represents the differential cost of responding in a higher category (compared to the one below). As one can see, $X_5 = X_1 + X_2 + X_3$ and thus also in the item and logit main-effects model, the predictor matrix contains linear dependencies so that not all parameters of these models can be estimated separately.

- Exercise 3.4

$$\begin{aligned}\Pr(Y_{pi} = 2) &= \Pr(V_{pi} > \beta_{i2}) = \Pr(O_{pi} + \theta_p > \beta_{i2}) \\ &= 1 - \Pr(O_{pi} \leq \beta_{i2} - \theta_p) = 1 - \Phi(\beta_{i2} - \theta_p) \\ &= \Phi(\theta_p - \beta_{i2}),\end{aligned}$$

where V_{pi} is distributed normally with mean θ_p and variance 1.

$$\begin{aligned}\Pr(Y_{pi} \geq 1) &= \Phi(\theta_p - \beta_{i1}), \\ \Pr(Y_{pi} \geq 2) &= \Phi(\theta_p - \beta_{i2}).\end{aligned} \tag{1}$$

- Exercise 3.5
In a model without person predictors, θ_p is distributed normal with mean zero and some variance. Then person predictors are brought into the model, with a mean that differs from zero. Since the random effect ϵ_p still has mean zero, the item parameters must have undergone a shift to compensate for the larger person part.

- Exercise 3.6
A LR test is needed because the variable consists of two predictors and the hypothesis that they are simultanuously zero has to be tested; this is not possible with a Wald test.

Table 1: Estimated regression coefficients for the PCM with person and (logit-specific) item predictors (verbal aggression data).

Predictor	Coefficient	Estimate	SE
Trait anger	ϑ_1	0.05	0.01
Gender	ϑ_2	0.27	0.12
Intercept	δ_{10}	1.56	0.23
	δ_{20}	1.98	0.23
Behavior Mode (doing)	δ_{11}	1.01	0.10
	δ_{21}	0.37	0.07
Type of Situation (other-blame)	δ_{12}	-0.96	0.10
	δ_{22}	-0.99	0.08
Type of Behavior (cursing)	δ_{13}	-1.07	0.13
	δ_{23}	-0.98	0.10
Type of Behavior (scolding)	δ_{14}	-0.38	0.12
	δ_{24}	-0.53	0.10
Mode $\times$ Type (cursing), L_1	$\delta_{1,13}$	-0.66	0.14
Mode $\times$ Type (scolding), L_1	$\delta_{1,14}$	-0.71	0.14
Situation $\times$ Type (cursing), L_1	$\delta_{1,23}$	0.58	0.14
Situation $\times$ Type (scolding), L_1	$\delta_{1,24}$	0.21	0.14

- Exercise 3.7
  With all pairwise interactions: Deviance= 766 $AIC = 812$ $BIC = 899$

  Excluding interactions between Behavior Mode and Type of Situation on both levels and interactions with Behavior Type on the second level gives the following fit statistics: Deviance= 771 $AIC = 805$ $BIC = 869$ and parameter estimates in Table 1.

  Interpretation of one of the interactions: Change in odds of responding perhaps rather than no (only interactions for the first logit), when going from shouting to curse for wanting and everything else constant = $e^{1.07} \approx 3$, but for doing $e^{1.07+0.66} \approx 6$.

- Exercise 3.8
  Software code:
  PROC NLMIXED data=aggression_poly method = gauss technique=newrap noad qpoints=20;
  TITLE2 'Rating Scale Model with person and item predictors';
  PARMS gamma1-gamma2=0 lambda=0 sd=1.5 alpha=0 beta1-beta4=0;
  theta=gamma1*traitang+gamma2*sex+int*th;
  eta1=theta-(alpha*int+beta1*ddo+beta2*self+beta3*curse+beta4*scold);
  eta2=theta-(alpha*int+beta1*ddo+beta2*self+beta3*curse+beta4*scold)-lambda; exp1=exp(eta1);

```
exp2=exp(eta1+eta2);
denom=1+exp1+exp2;
if (y=0) then p=1/denom;
else if (y=1) then p=exp1/denom;
else if (y=2) then p=exp2/denom;
if (p>1e-8) then ll=log(p);
else ll=−1e100;
MODEL y ~ general(ll);
RANDOM th ~ normal(0,sd**2) subject=pp;
ESTIMATE 'sd**2' sd**2;
RUN;
```

Results:
Deviance= 831 $AIC = 851$ $BIC = 889$ Estimates: $\hat{\gamma}_1 = 0.05(0.01)$ (Trait Anger)
$\hat{\gamma}_2 = 0.27(0.12)$ (Gender)
Intercept : 1.73(0.21)
Behavior Mode : 0.46(0.04)
Situation Type : −0.81(0.04)
Behavior Type : −1.37(0.05), −0.70(0.05)
$\hat{\sigma} = 0.90(0.05)$

For example, for Behavior Mode, the interpretation is as follows: Holding everything else constant, going from wanting to doing, decreases the odds of responding "perhaps" ("yes") rather than "no" ("perhaps") by about 35% ($\exp(-0.46) \approx 0.65$, 95% CI from 0.6 to 0.7).