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Solutions to the exercises of Chapter 10

- Exercise 10.1
  The challenge in this exercise is that you must see that a Rasch model has to be fit to the replicated data sets too in order to be able to compute the Q_3 index.

- Exercise 10.2
  Call δ_2 the pairwise interaction parameter and δ_3 the three-way interaction parameter. Then,

$$\boldsymbol{W}_p = \begin{pmatrix} 0 & 0 \\ y_{p1} & 0 \\ y_{p1}+y_{p2} & y_{p1}y_{p2} \\ y_{p1}+y_{p2}+y_{p3} & y_{p1}y_{p2}+y_{p1}y_{p3}+y_{p2}y_{p3} \\ y_{p1}+y_{p2}+y_{p3}+y_{p4} & \begin{array}{c} y_{p1}y_{p2}+y_{p1}y_{p3}+y_{p1}y_{p4} \\ +y_{p2}y_{p3}+y_{p2}y_{p3}+ \\ y_{p2}y_{p4}+y_{p3}y_{p4} \end{array} \end{pmatrix}.$$

  Define $t_1 = \sum_{j=1}^{i-1} y_{pj}$ and $t_2 = \sum_{\substack{j=1 \\ j<k}}^{i-2} \sum_{k=1}^{i-1} y_{pj}y_{pk}$. Interpretation: Holding everything else constant, the odds of giving a 1-response change with factor $e^{\delta_2(t_1-t_1^*)}$ if t_1 instead of t_1^* previous successes are achieved. For δ_3 a similar interpretation holds, but then for an increase in previous number of pairs of successes. Note however that the "holding everything constant" requirement is somewhat artificial here because if t_1 changes, t_2 is likely to change as well (but not automatically).

- Exercise 10.3
  For two items:

$$\begin{aligned} \Pr(Y_1=1) &= \sum_{y=0}^{1} \Pr(Y_1=1, Y_2=y) \\ &= \frac{\exp(\theta_p-\beta_1)+\exp(2\theta_p-\beta_1-\beta_2+\delta)}{1+\exp(\theta_p-\beta_1)+\exp(\theta_p-\beta_2)+\exp(2\theta_p-\beta_1-\beta_2+\delta)} \end{aligned}$$

- Exercise 10.4

$$\begin{aligned}
\Pr(Y_1 = 1, Y_2 = y) &= \int \Pr(Y_1 = 1, Y_2 = y|\theta_p) dF(\theta_p) \\
&= \int \Pr(Y_1 = 1|\theta_p)\Pr(Y_2 = y|\theta_p) dF(\theta_p) \\
\Pr(Y_1 = 1) &= \sum_{y=0}^{1} \Pr(Y_1 = 1, Y_2 = y) \\
&= \int \Pr(Y_1 = 1|\theta_p) dF(\theta_p)
\end{aligned}$$

- Exercise 10.5

$$R = \left(\begin{array}{ccc|ccc}
1 & \cdots & \rho_1 & \rho_2 & \cdots & \rho_2 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\rho_1 & \cdots & 1 & \rho_2 & \cdots & \rho_2 \\
\hline
\rho_2 & \cdots & \rho_2 & 1 & \cdots & \rho_3 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\rho_2 & \cdots & \rho_2 & \rho_3 & \cdots & 1
\end{array}\right),$$

with $\rho_1 = \frac{\sigma^2+\tau^2}{\sigma^2+\tau^2+1}$, $\rho_2 = \frac{\sigma^2}{\sqrt{\sigma^2+\tau^2+1}\sqrt{\sigma^2+1}}$, and $\rho_3 = \frac{\sigma^2}{\sigma^2+1}$

- Exercise 10.6

$$\begin{aligned}
\Pr(Y_1 = y_1) &= \frac{\exp(y_1(\theta - \beta_1))}{1 + \exp(\theta - \beta_1)} \\
\Pr(Y_2 = y_2|y_1) &= \frac{\exp(y_2(\theta - \beta_2 + y_1\delta_1))}{1 + \exp(\theta - \beta_2 + y_1\delta_1)}
\end{aligned}$$

and

$$\Pr(Y_3 = y_3, Y_4 = y_4|y_1, y_2) = \frac{\exp(y_3(\theta - \beta_3 + s\delta_1) + y_4(\theta - \beta_4 + s\delta_1) + y_3y_4\delta_2)}{1 + \exp(\theta - \beta_3 + s\delta_1) + \exp(\theta - \beta_4 + s\delta_1) + \exp(2\theta - \beta_3 - \beta_4 + 2s\delta_1 + \delta_2)}$$

where $s = y_1 + y_2$.

- Exercise 10.7
  Design matrix for the normal child:

$$\boldsymbol{X} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & y_1 & 0 \\
0 & 0 & 1 & 0 & 0 & y_1 + y_2 & 0 \\
0 & 0 & 0 & 1 & 0 & y_1 + y + 2 + y_3 & 0 \\
0 & 0 & 0 & 0 & 1 & y_1 + y_2 + y_3 + y_4 & 0
\end{pmatrix}$$

Design matrix for the disabled child:

$$\boldsymbol{X} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & y_1 \\ 0 & 0 & 1 & 0 & 0 & 0 & y_1 + y_2 \\ 0 & 0 & 0 & 1 & 0 & 0 & y_1 + y_2 + y_3 \\ 0 & 0 & 0 & 0 & 1 & 0 & y_1 + y_2 + y_3 + y_4 \end{pmatrix}$$

- Exercise 10.8
  The model can be constructed by simply adding θ_{p1} to the exponents in all cumulative probability equations pertaining to the items involved in the testlet.